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Introduction : Three-Particle Systems

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A three particle system might be called a-few-body system. However, in many cases it displays most of the essential properties of a many particle system of the same nature. So, as an introduction, I would like to remind you of the investigations of three particle systems, which played important roles in elucidating fundamental properties or basic mathematical method of nonlinear problems.

A three particle periodic system is essentially a system of two degrees of freedom because one of the three coordinates is the so-called cyclic coordinate and therefore can be eliminated. So, I will speak also of systems of two degrees of freedom.

I shall introduce the problems in a chronological order.

0) Historically, the three-body problems of the celestial bodies under gravitational forces is the most famous problem. But this is out of scope of the present discussion.

1) In 1972, J.Ford[1)] pointed out that the three particle system with cubic nonlinearity in the interaction potential was equivalent to the so-called Hénon-Heiles system[2)], which showed stochastic behavior when the energy was raised above a certain critical value.

The Hamiltonian of this system may be written as

$$H = \frac{1}{2}[(P_1^2+P_2^2+p_3^2)+(Q_1-Q_3)^2+(Q_2-Q_1)^2+(Q_3-Q_2)^2] - \frac{1}{3}\alpha[(Q_1-Q_3)^3+(Q_2-Q_1)^3+(Q_3-Q_2)^3]$$

We introduce the normal coordinates and momenta (ζ_j,η_j) of the system with $\alpha=0$, the linear system. We find then that the Hamiltonian is transformed into

$$H = \frac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2+3\zeta_1^2+3\zeta_2^2) + \frac{3\alpha}{2^{1/2}}(\zeta_2\zeta_1^2-\frac{1}{3}\zeta_2^3)$$

Since η_3 is a constant of motion (uniform circulation of the ring), we may drop the corresponding term from H. By the rescaling

$$t \to t/3^{1/2} \quad , \quad H \to 6H/\alpha^2$$

$$\zeta_2 = (2^{1/2}/\alpha)q_2 \ , \ \zeta_1 = (2^{1/2}/\alpha)q_1$$

the Hamiltonian is transformed into

$$H = \frac{1}{2}(p_1^2+p_2^2)+ \frac{1}{2}(q_1^2+q_2^2)+ q_1^2 q_2 - \frac{1}{3}q_2^3 \quad ,$$

which is just the Hénon-Heiles system. This is originally considered as a model for the motion of stars in a simplified galaxy. It is known that this system shows stochastic behavior when the energy is raised above a certain critical value. Therefore the nonlinear lattice with cubic nonlinearity of the interaction potential is also non-integrable.

2) In 1971 N.Saito treated numerically the system of two particles with exponential interaction using fixed end boundary conditions[3]. It was shown that the system showed quite smooth mapping of trajectories on the Poincaré surface of section indicating that it is integrable. Independently J.Ford[4] showed in detail that the three particle periodic system with exponential interaction was integrable, and this paper led H.Flaschka to the analytical works. The system J.Ford showed integrable is

$$H = \frac{1}{2}(P_1^2+P_2^2+P_3^2) + e^{-(Q_1-Q_3)} + e^{-(Q_2-Q_1)} + e^{-(Q_3-Q_2)} - 3 .$$

3) In 1973, M.Kac[5] showed analytically that the periodic system with the above Hamiltonian is indeed integrable in the sense of Jacobi's inversion problem and with van Moerbeke he extended the treatment to periodic system with the exponential interaction composed of arbitrary number of particles[6]. Independently, a concrete solution to this problem was given by E.Date and S.Tanaka[7].

Since the calculation for a three particle system looks simple and illuminating, I will show in some detail a canonical transformation in the following.

The system is chracterized by the elements

$$a_0 = a_3 = \frac{1}{2} e^{-(Q - Q)/2}$$

$$a_1 = \frac{1}{2} e^{-(Q - Q)/2}$$

$$a_2 = \frac{1}{2} e^{-(Q - Q)/2}$$

$$b_0 = b_3 = \frac{1}{2} P_0$$

$$b_1 = \frac{1}{2} P_1$$

$$b_2 = \frac{1}{2} P_2$$

The discriminant

$$\Delta(\lambda) = \frac{1}{a_0 a_1 a_2}(\lambda-b_1)(\lambda-b_2)(\lambda-b_0) - \frac{a_2}{a_0 a_1}(\lambda-b_1) - \frac{a_0}{a_1 a_2}(\lambda-b_2) - \frac{a_1}{a_2 a_0}(\lambda-b_0)$$

is shown to be independent of time, so that the curve $\Delta(\lambda)$ vs. λ is fixed by the initial condition. The roots of

$$\Delta(\lambda)^2 - 4 = 0$$

are called the spectra, $\lambda_1 < \lambda_2 \leqq \lambda_3 < \lambda_4 \leqq \lambda_5 < \lambda_6$.

The auxiliary spectra are introduced as

$$\mu_1 = \frac{b_2 + b_0 - \sqrt{(b_2 - b_0)^2 + 4a_2^2}}{2}$$

$$\mu_2 = \frac{b_2 + b_0 + \sqrt{(b_2 - b_0)^2 + 4a_2^2}}{2}$$

We see that

$$\lambda_2 \leqq \mu_1 \leqq \lambda_3$$

$$\lambda_4 \leqq \mu_2 \leqq \lambda_5 \quad .$$

If we let

$$\Delta_1 = \Delta(\mu_1) \quad , \Delta_2 = \Delta(\mu_2)$$

we can show that the Poisson brackets turn out to be

$$[\mu_1, \Delta_1] = \frac{1}{2}\sqrt{\Delta_1^2 - 4}$$

$$[\mu_2, \Delta_2] = \frac{1}{2}\sqrt{\Delta_2^2 - 4}$$

$$[\mu_1, \mu_2] = [\Delta_1, \Delta_2] = [\mu_1, \Delta_2] = [\mu_2, \Delta_1] = 0$$

In view of the general relation $[q, f(p)] = f'(p)$, we see that

$$\nu_1 = 2\cosh^{-1}\frac{\Delta_1}{2} = 2\log\frac{\Delta_1+\sqrt{\Delta_1^2-4}}{2}$$

is the momentum conjugate to μ_1, and

$$\nu_2 = 2\log\left|\frac{\Delta_2-\sqrt{\Delta_2^2-4}}{2}\right|$$

is the momentum conjugate to μ_2. Thus

$$[\mu_1, \nu_1] = [\mu_2, \nu_2] = 1$$

and other Poisson brackets vanish.

It turns out that

$$H = 2(b_1^2+b_2^2+b_0^2)+4(a_1^2+a_2^2+a_0^2)$$

$$= 4\frac{-\frac{1}{8}(\Delta_1-\Delta_2)+\mu_1^3+\mu_2^3}{\mu_1-\mu_2} - 2(\mu_1+\mu_2)b+2b^2$$

where

$$b \equiv b_0+b_1+b_2 = \text{const.}$$

We may choose the coordinate such that $b\equiv 0$.

Then we have

$$H = 4\,\frac{-\frac{1}{4}[\cosh\frac{\nu_1}{2} - \cosh\frac{\nu_2}{2}] + \mu_1^3+\mu_2^3}{\mu_1-\mu_2}$$

The equations of motion for the auxiliary spectra are

$$\dot{\mu}_1 = \frac{\partial H}{\partial \nu_1} = \frac{1}{2}\,\frac{\sinh \nu_1/2}{\mu_1-\mu_2} = \pm\frac{1}{4}\,\frac{\sqrt{\Delta_1^2-4}}{\mu_1-\mu_2}$$

$$\dot{\mu}_2 = \frac{\partial H}{\partial \nu_2} = -\frac{1}{2}\frac{\sinh \nu_2/2}{\mu_1-\mu_2} = \pm\frac{1}{4}\frac{\sqrt{\Delta_2^2-4}}{\mu_1-\mu_2}$$

μ_1 and μ_2 oscillate between (λ_2, λ_3) and (λ_4, λ_5).

We may introduce the action variables, for example by

$$J_1 = \oint \nu_1 d\mu_1 = 4\int_{\lambda_2}^{\lambda_3} \log \frac{\Delta(\mu)+\sqrt{\Delta^2(\mu)-4}}{2}\, d\mu$$

General theory of such transformations was extended by Flaschka and McLaughlin, and by others[8].

4) In 1977 G.Casati and J.Ford[9] showed that a two particle system of unequal masses joined by exponential interaction (both ends fixed) exhibits stochastic behavior when the energy exceeds critical value depending on the mass ratio. This seems to be a new and very interesting phenomena which is waiting for an analytic explanation. I have no special idea now, but I would like to present two equivalent systems in this respect.

One is the dual system

$$H = \frac{(s_1-s_2)^2}{2m_1} + \frac{s_2^2}{2m_2} + e^{-r_1} + e^{-r_2} + e^{-r_3} \quad , \qquad (r_3 = -(r_1+r_2))$$

with the equations of motion

$$\frac{d}{dt}\log(1+\dot{s}_1) = \frac{s_2-s_1}{m_1}$$

$$\frac{d}{dt}\log(1+\dot{s}_2) = \frac{s_1-s_2}{m_1} - \frac{s_2}{m_2} \quad .$$

The other is the equation for $q_2' = dq_2/dq_1$. This method follows Whittaker's Analitical Dynamics[10]. We note the Lagrangian

$$L = \frac{1}{2}(m_1\dot{q}_1^2 + m_2\dot{q}_2^2) - (e^{-q_1} + e^{-(q_2-q_1)} + e^{q_2} - 3) \quad .$$

The total energy is

$$E = \frac{1}{2}\dot{q}_1^2(m_1+m_2q_2'^2) + (e^{-q_1} + e^{-(q_2-q_1)} + e^{q_2} - 3)$$

Following Whittaker we consider $\dot{q}_1=\dot{q}_1(E,q_2',q_2,q_1)$ as a function of E, q_2', q_2 and q_1, and set up

$$\frac{\partial L}{\partial \dot{q}_1} = L'(E,q_2',q_2,q_1) \quad .$$

Then we have the equation of motion

$$\frac{d}{dq_1}\frac{\partial L'}{\partial q_2'} - \frac{\partial L'}{\partial q_2} = 0$$

which turns out to be

$$\frac{d}{dq_1}\frac{q_2'}{\sqrt{m_1+m_2q_2'^2}} + \frac{G+Hq_2'}{F}\frac{q_2'}{\sqrt{m_1+m_2q_2'^2}} - \frac{H}{F}\frac{\sqrt{m_1+m_2q_2'^2}}{m_2} = 0$$

where

$$F = E - (e^{-q_1} + e^{-(q_2-q_1)} + e^{q_2} - 3)$$

$$G = \frac{1}{2}\{ e^{-q_1} - e^{-(q_2-q_1)} \}$$

$$H = \frac{1}{2}\{ e^{-(q_2-q_1)} - e^{q_2} \} \quad .$$

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